Capital Gains and Their Sources

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INTRODUCTION AND SUMMARY

Capital gains occupy a somewhat anomalous position in economic theory. They do not fit neatly into any of the usual classifications of income as an agreed upon, relatively risk free, market determined return to the supplier of a resource such as land, labor, capital or technology. They are, in fact, for the most part a kind of income once removed, a change in the present value of an income stream rather than the stream itself.\(^1\) This ambiguity about the sources and the role of gains is reflected both in their treatment in aggregate demand theory, where they have yet to find a suitable place, and in their tax treatment by governments.\(^2\)

The general objective of this study is to analyze both theoretically and empirically the nature of capital gains on income earning assets and other forms of income from capital.\(^3\)

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\(^1\) There are, of course, changes in the capital value of assets that provide no income stream, such as stamps, but our initial concern will be only with the present value of income earning financial assets.

\(^2\) In many parts of the world (Hong Kong and, until quite recently, France) capital gains escape taxation entirely. In other places such as the United States, they are defined rather arbitrarily as either short-term or long-term and taxed as ordinary income in one case and at preferential rates in the other.

\(^3\) Since gains are, in effect, income once removed, taxation of the income stream and taxation of a gain are closely linked.
Initially, the major thrust will be to attempt to determine the role played in generating capital gains by four important factors: retained earnings, inflation, changes in the discount rate, and unexpected changes in earnings. If the influence of each can be isolated, the information will help shed light on some questions that are intimately linked with policy problems of various kinds. For example, it may be possible to determine whether the rate of return on retained earnings exceeds the cost of capital for individual firms. If the return exceeds the cost in the majority of cases, it will help to explain the results of recent studies indicating that the return on the outside capital exceeds that on internal funds. The usual interpretation of this finding is that firms are more efficient at investing externally raised funds than retained earnings. A finding that returns exceed costs for internal funds would lend support to the alternative hypothesis that firms invest internally generated funds efficiently and turn to external capital markets only when internal funds are inadequate to fund investments of higher than average profitability. In addition, it may be feasible to evaluate the investment decisions of large firms with potential monopoly power to determine if their behavior or degree of efficiency seems to differ from that of smaller, less dominant firms.

The degree to which stocks have proven to be a hedge
against inflation also has policy implications. If capital markets are to function efficiently during inflationary periods, stocks should be a good hedge against inflation. Otherwise investors will forsake them and move into other assets they think offer better guarantees of purchasing power. If, historically, stocks have not provided insurance against inflation it may well be partially due to government policies governing depreciation, taxes, and investment incentives. Identifying stocks that have been good inflation hedges and those that have been poor ones may provide some policy guidance.

Another example of a question into which some insight may be offered is that of how investors evaluate new investments made by established firms. Do they, in contemplating the projected earnings stream from a new investment tend to use the firm's current average cost of capital and earnings growth rate in arriving at a discounted value for the project, or do they use a marginal cost of capital and growth rate more appropriate to the individual project itself, given its apparent risk and potential cash flow.

The study will be divided into three basic parts. The first will involve setting up and describing a model that relates capital gains on income earning assets to the sources of those gains and defines the theoretical relationship between the gains and their sources. The second will
include estimating the parameters of the model for all the individual stocks for which data are available on the COMPUSTAT tapes and evaluating the results thus obtained. The third will be more theoretical in nature and will examine more fully the nature of capital gains as a source of income and analyze the potential impact of various alternative tax treatments of capital gains and other income from capital upon the equitable distribution of income, investment incentives, and the cost of capital for firms. In particular, a comparison will be made between a consumption tax and an income tax and the proper role of a capital gains tax under each regime. The results obtained from the empirical work will be analyzed for their implications for tax policy. In addition, the similarities and differences between gains on income earning assets and non-income earning assets will be examined as will the consequences for tax policy of the differences. This initial paper, however, is concerned only with the first part of the study, and includes a derivation of the model relating capital gains to their sources and a discussion of the potential usefulness of the model and of the estimates of its parameters.

The basic stock valuation model from which are derived the linkages between capital gains and their causes is the familiar discounted cash flow model:

\[ V = \frac{aE}{s-g} \]
where $V$ is the price of a share, $E$ is the current earnings per share, $a$ is the dividend payout ratio, $\beta$ is the discount rate applied by the stock market to the firm's earnings streams and $g$ is the expected rate of growth. From this equation we can develop the following relationship:  

$$ \delta V = \frac{\partial V}{\partial a}\delta a + \frac{\partial V}{\beta (\beta - g)}\delta (\beta - g) + \frac{\partial V}{\partial P}\delta P + \frac{\partial V}{\partial E}\delta E. $$

The left side of the equation is the change in capital value. In order, the right hand terms are the effects of: (1) retained earnings, (2) changes in the discount rate and expected growth rate for the stock, (3) changes in the expected rate of inflation, (4) changes in the price level, and (5) unforeseen earnings changes.

These factors are the most important determinants of changes in the prices of shares. There are four fundamental questions that must be posed concerning their effects on capital values: (1) What are the theoretical effects of each of the factors on share prices, (2) What are the effects actually observed, (3) Can the differences be reconciled, explained, and interpreted, and (4) What is the importance of each one relative to the others. To answer the questions, we...
can cast equation (B) as a regression equation with
\( y = \text{capital gains} \) and the right hand terms written as a series
of \( X_i \). Items (2) and (3) above, i.e., the discount rate
and the expected rate of inflation, must be included (at
least initially) as part of the same variable since their
effects are similar and as a practical matter they are virtu-
ally indistinguishable. When equation (B) is rewritten as
a regression equation we have:

\[
(C) \quad y = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \epsilon
\]

where \( X_1 = \text{the retained earnings variable}, \ X_2 = \text{the discount}
rate/growth rate/inflation rate variable, \ X_3 = \text{the price}
level variable, and \( X_4 = \text{the unforeseen earnings variable} \).

The theoretical values for the regression parameters
can be obtained by examining equations (B) and (C). The first
variable, \( X_1 \), is retained earnings and corresponds to \( \frac{\partial V}{\partial B} \)
in equation (B). Thus, \( \alpha_1 \) is an estimate of the partial \( \frac{\partial V}{\partial B} \),
i.e., the ratio of the change in share price to an increase
in retained earnings. As we shall demonstrate later, this
ratio should be 1 if the firm's expected rate of return
on investment is equal to the market discount rate or cost
of capital. If this condition does not hold, the most likely
positive range for \( \alpha_1 \) is between 0 and 4 with values over
1 occurring when the rate of return exceeds the discount
rate. Negative values for \( \alpha_1 \) are not ruled out, however. A
negative value would simply imply the firm lost money on the reinvested earnings. Thus, estimates of \( a_1 \) for individual stocks will provide inferences about the relative values of the rate of return and the discount rate for the firm. In addition, cross-section analyses will be done to attempt to determine how the ratios have varied over time.

The second variable, \( X_2 \), takes into account the effects of changes in the discount rate, the growth rate, and the expected rate of inflation. As will be demonstrated later, all three operate in an identical fashion on capital values. The variable itself is the entire partial derivative, and as a result, the theoretical value of \( a_2 \) is 1.

The third variable, \( X_3 \), reflects changes in the price level. It is essential to distinguish changes in the price level from changes in the expected rate of inflation. Changes in the price level will result in realized changes in costs, selling prices and profits quite independently of what may happen to the expected rate of inflation. If the firm can change its prices as its costs change without affecting demand for its product, its profits will rise with inflation and the stock's price should reflect the increased profitability. The variable used is the entire partial, thus the coefficient, \( a_3 \), should be a direct measure of the firm's ability to hedge against inflation. If the firm is just able to offset inflation, it should have a value of 1 for
the price level coefficient. If it more than offsets inflation the value should be greater than 1. It is not inconceivable that the value should be negative. This would occur if a firm's profits fall when the price level increases.

The fourth variable is one which measures the difference between an expected profit level and that which actually occurred. It is always debatable how one should measure expectations but we shall initially use some naive projections and compare these projections to realized profits. The variable is defined as the entire partial derivative but there are no a priori values that we can assign to the coefficient \( a_4 \) other than that it should probably be positive and less than 1. If expectations were estimated fairly reliably and any realized differences from those expectations were assumed by the market to be permanent the coefficient should be approximately 1. On the other hand, if realized differences were thought by the market to be totally random and had no effect on expectations, the coefficient should be zero. Any combination of the two would yield a value between these two limits.

Over all, the model can be characterized as a financial model of capital gains and as such can be expected to explain only a portion of the gains for any individual firm. It is to be expected that this portion may vary widely from firm to firm. The elements not included in the model are what one
might call the real factors that influence firms' profits and stock prices, namely those things associated with the demand for and prices of a firm's output. The more widely these elements fluctuate, the greater will be the influence of factors not included in the model on the firm's share price. Thus, the model may well explain only a small part of the variation in share price of a firm whose sales, prices, and profits are highly cyclical but explain a large part of the price variation for firms in stable industries. The coefficient of determination obtained from the regression program will give an estimate of how important those non-financial factors are for each firm.

In addition to indicating the degree to which changes in individual independent variables are related to changes in stock price, the regression results can be used to estimate the relative importance of each in causing capital gains over different periods. In the estimated equation, the entire gain is allocated among the various influences, including the random disturbance which is a measure of the influence of real factors. By evaluating the equation at the mean of the absolute values for each variable, it will be possible to estimate the average proportional contribution of each to the average gain. In addition to their intrinsic interest, such estimates could prove useful in predicting stockholder behavior. For example, if it were to turn out that on average retained earnings were responsible for only 15 percent of the capital
gains, one could argue that shareholders are likely to be less concerned about whether investment policies lead to a one-to-one or greater reflection of retained earnings in gains than they would be if retained earnings contributed 85 percent of gains. The relative importance of various variables might also give some clues as to how stock analysts should allocate their time.

One question that has received a certain amount of coverage in the literature is whether or not firms invest the earnings they retain as wisely and as profitably as they do the funds they raise from outside sources through borrowing on stock issues.¹ A simplified statement of the consensus reached by studies of this question is that return on outside funds is higher than that on retained earnings. None of the studies, however, has attempted to determine whether the return on retained earnings is less than or greater than the cost of capital. This is a far more important question than whether the return on external funds exceeds that on internal

funds (for which there are many plausible explanations). Interpretation of the results in this study (particularly as described above) may provide some insights about the relationships between the cost of capital and the rate of return on investment for various firms. It may be possible to identify the characteristics of firms with higher as well as lower returns relative to the cost of capital. One obvious characteristic to be examined would be whether or not the firm was raising funds externally.

All the results of the study will be analyzed within a framework constructed of the various characteristics of the firms in the sample in order to determine if there are any particular patterns than can be observed. One of the most important characteristics will be size of the firm. We can examine the role of size as an important factor in determining whether a firm's stock is a good hedge against inflation or whether retained earnings show up as capital gains or whether the firm seems to earn a return greater or less than its cost of capital.

In addition to size, there are a number of interesting characteristics that can be examined including:

1. The variety of goods produced by the firm, i.e., its degree of diversification.
2. The financial structure of the firm and the average length of its debt.
3. The percentage of business done overseas.
4. The capital intensity, e.g., the capital/labor ratios.
5. Age of the firm.
6. The rate of growth.
7. Rate of capital turnover, e.g., ratio of depreciation to total value of fixed assets.
8. The payout ratio.

In what follows we discuss in more detail the points outlined in this introduction and discuss the theoretical aspects of the study. The main body of the paper is followed by an appendix in which the basic equations are derived and the regression models discussed.
The Capital Gains Model

The model most commonly used in analyzing the economic value of securities is the Discounted Cash Flow (DCF) model which defines the price of a security as the present value of the expected cash stream that the security represents a claim to. For an equity this cash stream is composed of the future dividends a stockholder can expect to receive over the period of time the stock will be outstanding. A stock is a perpetuity, however, and unless the stockholder expects the company to go out of business the time horizon stretches to infinity.

The income stream has certain characteristics which determine for the investor the degree of risk associated with it. This risk is taken into account by the market participants in assigning to the income stream a discount rate which includes the risk premium investors require if they are to purchase that particular stream of income. Another attribute that investors associate with the income stream is growth. Growth can either be positive, negative or zero, but, in general, if earnings are retained and invested, it is logical to believe that earnings growth will be positive as the firm's capacity expands. These relationships among security price, dividends, risk and growth can be expressed as a rather simple equation.

If we let \( V \) be the price of a share, \( D \) be the current dividend rate, \( \delta \) be the rate of time discount applied by
the market to cash flows of equivalent riskiness and \( g \) be the expected rate of growth of the dividend, the formula is:

\[
V = \frac{D}{\beta - g}
\]

If the dividend is defined as the product of the payout ratio, \( a \), and the level of earnings, \( E \), and earnings are the rate of return, \( r \), times book value, \( B \), the formula can be written either of two ways:

\[
(2) \quad V = \frac{aE}{\beta - g} = \frac{arB}{\beta - g}
\]

We can further refine the formula to define more completely the price as a function of accounting variables by observing that if the rate of return on retained earnings is equal to that on the current capital stock, the rate of growth of the income stream will be:

\[
(3) \quad g = \frac{R}{B} = (1 - a)r
\]

where \( R \) is retained earnings and \( B \) is the book value of the firm.\(^1\)

\(^1\)If the rate of return on old and new investments is \( r \), \( E = rB \) and \( \Delta E = rR \). The rate of growth is:

\[
g = \frac{\Delta E}{E} = \frac{R}{B}
\]

If we set \( R = (1 - a)E = (1 - a)rB \) we have \( g = (1 - a)r \). Growth is the product of the retained earnings ratio and the rate of return on capital.
Since retained earnings are total earnings less the dividend, \( R = (1 - a)E \), equation (2) becomes:

\[
V = \frac{\frac{aE}{\beta}}{(1-a)E} = \frac{aE}{\beta} \frac{E}{(1-a)r} = \frac{arB}{\beta - (1-a)r}
\]

We shall utilize later the formulation of equation (4) but for the moment let us return to the basic equation:

\[
V = \frac{\frac{aE}{\beta}}{(1-a)E}
\]

This equation is nothing more than a variant of the present value formula for a perpetual income stream, \( V = \frac{C}{r} \). A fixed income security with no maturation date, such as a preferred stock, has a known coupon, \( C \), that will be paid periodically over the foreseeable future. The value of such a perpetuity is determined by dividing the coupon by the appropriate market rate of interest. If the company remains in business and there is no question about the safety of the coupon, the only thing that affects the value of the preferred stock is a change in the market rate of a interest. Although the basic formula relating the value of a common stock to its expected earnings is very similar to the formula for a fixed income perpetuity, the valuation process is fundamentally different because the income stream for a common stock is not only uncertain, it is also affected by decisions about re-investment of earnings.

Since a primary objective of the study is to measure the
relative importance of the financial factors that influence
the prices of common stocks and result in capital gains or
losses, it is essential to identify the theoretical rela-
tionship between a change in each of the factors and the
resultant change in stock price.\(^1\) We can use as our starting
point equation (2):

\[
V = \frac{\alpha E}{\beta - g} = \frac{arB}{\beta - g} .
\]

Taking the total differential of \(V\) we have:

\[
dV = \frac{3V}{3B} dB + \frac{3V}{3(\beta - g)} d(\beta - g) + \frac{3V}{3P} dp + \frac{3V}{3P} dp + \frac{3V}{3E} dE .
\]

The right hand side of equation (5) includes changes in the
following: retained earnings, the discount rate, the rate of
growth, the expected rate of inflation, the price level, and
the expected income stream. In the remainder of the paper
we shall look at each of these variables separately.

1. **Retained earnings.** In equation (3) the effects of retained
earnings are included in the first term, \(\frac{3V}{3B} dB\), where \(dB\),
the change in book value is equivalent to retained earnings.
It is evident that the reinvestment of earnings by the firm
should lead to an increase in the value of the firm and the

\(^1\)In Appendix A we derive these relationships mathematically.
The discussion in this section is primarily verbal. We accept \(\alpha\) as a parameter.
price of a share. What are not evident are the conditions under which there should be a one-to-one relationship between retained earnings and increased share price. The one situation in which a given amount of retained earnings will result in an identical increase in share price is when the market discount rate is equal to the rate of return on invested capital which in turn is equal to the rate of return on reinvested earnings.

When one departs from these rather strict conditions, however, the market value of retained earnings depends upon the relationship between the rate of return and the market discount rate as well as upon the expected rate of growth applied by the market to earnings from new investments and the dividend payout ratio. In the following sections we shall demonstrate how these variables interact to determine the relationship between a dollar's worth of retained earnings and the amount of capital gain that results. We shall see that under quite reasonable assumptions the ratio of gain to retained earnings may range between zero and four and may even take on negative values. Before we do so, however, it is useful to spell out the conditions under which gains should just equal earnings. The conditions differ as a function of how the expected rate of growth of earnings from new investments is set.

There are two possible ways in which the expected rate of growth could be determined by the market. (1) The market
could continue to apply its formerly estimated growth rate, 
\[ g = (1-\alpha)r_0 \], in evaluating the new stream of earnings regardless of how the expected return on new investment, \( r' \), differs from the historical return, \( r_0 \). (2) The market could apply a new estimated growth rate, \( g' = (1-\alpha)r' \), that is a function of the expected return on new investment. It applies this growth rate, however, only to the income stream attached to the new investment. We shall refer to the first situation as Model I and the second situation as Model II in the discussion that follows.

a. Conditions for equality of retained earnings and capital gains

**Model I**

If the market uses a historical growth rate, \( g = (1-\alpha)r_0 \), in evaluating earnings streams from reinvested earnings, there exists a special relationship which defines the conditions under which capital gains will be equal to retained earnings. The condition for equality of gains is that the discount rate (\( \delta \)) be equal to the return on existing capital (\( r_0 \)) plus the dividend payout ratio (\( \alpha \)) times the difference between the return on existing capital and that on new investment (\( r' \)). That is, gains equal retained earnings if:

\[ \delta = r_0 + \alpha (r - r_0) \]

These conditions are derived in Appendix A. They are valid if one simplifying (but realistic) assumption is made. That is, if the return on new investment, \( r' \), is different from the return on existing capital, \( r \), the difference holds only for one period. If this assumption holds, the projected rate of growth will not be affected.
The location of this function is illustrated in figure I. If the discount rate is below that defined by equation (6) the gain will exceed the retained earnings. Conversely, if the discount rate is higher, gains will be less than retained earnings.

The slope of the boundary line is equal to \( a \), the dividend payout ratio. As \( a \) approaches 1, the value of the market discount rate that assures equality between the capital gain and the retained earnings approaches the rate of return on new investment. On the other hand, as \( a \) approaches zero, the equalizing discount rate approaches the rate of return on existing capital.

Model II

A second, and perhaps more likely, possibility for evaluation of retained earnings is that the market evaluates these earnings using a growth rate that is a function of the return on new investment so that \( g' = (1-a)r_0' \). In this case, if the market discount rate is equal to the return on

\[
(6) \quad \beta = r_0 + a(r' - r_0)
\]

We have in this case

\[
V_1 = \frac{\alpha E}{\beta - (1-a)r} \quad V_2 = \frac{\alpha E}{\beta - (1-a)r} \frac{ar'R}{\beta (1-a)r'}
\]

\[
V = \frac{ar'R}{\beta - (1-a)r'} \frac{ar'R}{\beta - r' + ar'} \quad V < R \text{ as } B < r'
\]
\[ \beta = \text{market discount rate} \]

\[ (1-\alpha)r_0 \]

\[ r_0 = r' = \beta \]

Region where gain is less than retained earnings

\[ \beta = r_0 + \alpha (r' - r_0) \]

Region where capital gain exceeds retained earnings

\[ r_0 = \text{return on already invested capital} \]

\[ r' = \text{return on invested earnings} \]

Figure 1. Conditions for equality of capital gains and retained earnings - Evaluation Model I
new investment, the capital gain will equal the retained earnings. If the market discount rate is below the return on investment gains will exceed retained earnings. The converse will be true if the discount rate is above the rate of return. This function is illustrated in figure 2.

It should be fairly evident by now that the conditions which would lead to an observed one-to-one relationship between capital gains and retained earnings in each period or even over long periods (all other variables held constant or taken into account) may well not be present. A single observation of a firm's stock price and the other variables involved in equation (4) would find a particular value for \( r_0 \), the return on capital, generated by the firm's historical investment decisions and current operations. The market discount rate, \( \delta \), might be considerably different from \( r_0 \). It might also be considerably different from \( r' \), the expected return on new investment and from \( r_0 + \alpha(r' - r_1) \). As a result, regardless of whether the model is used by the market to value reinvested earnings, retains earnings and capital gains would not be equal.

The most plausible case in which retained earnings and capital gains might be approximately equal is that in which the firm behaves in accordance with economic theory and uses the market discount rate as its criteria for investment. In this case it will invest up to the point where the return
Figure 2. Conditions for equality of capital gains and retained earnings — Model II
on the marginal investment equals the discount rate, i.e., \( \beta = r' \). If the market used Model II for evaluating reinvested earnings, capital gains would be equal to retained earnings.

On the other hand, if the firm equates \( \beta \) and \( r' \) but the market uses Model I to evaluate retained earnings, gains would exceed retained earnings if \( \beta < r_0 \) and be less than retained earnings if \( \beta > r_0 \). Equality would be achieved only at the single point where \( \beta = r_0 = r' \).

The parameter associated with the capital gains terms in the regression equation should permit an evaluation of whether or not capital gains have in fact exceeded retained earnings and by how much. It will give little indication, however, of which model of growth the market uses, unless one can estimate the individual parameters, \( r_0, \beta \) and \( r' \). The reasons become clear if we look at Figure 3. The line labeled \( \beta = r' \) is the function that indicates equality of gains and retained earnings if Model II is used by the market. The line labeled \( \beta = r_0 + a(r' - r_0) \) is the function that indicates equality of gains and retained earnings if Model I is used by the market. There is no way to know which model applies without knowledge of all three parameters.

Knowledge of which model is used by the market would be

\[ ^{1}\text{If these estimations can be done, and we shall attempt them, it will be possible to identify more completely the conditions that actually pertain for most stocks.} \]
Figure III. Conditions for equality of capital gains and retained earnings (Models I and II)
useful because if the market uses Model I, there is an area A where the marginal return on capital is below the cost of capital and below the historical average return but where reinvestment of retained earnings would lead to gains above the cost of the investment. Conversely, again using model I, there is another area B where the marginal return on capital is above the cost of capital and above the historical average. In this area, however, the investment of retained earnings would lead to gains less than the amount invested. It would be interesting to know the extent to which these paradoxical situations might exist or persist.

b. Determinants of the Capital Gains/Retained Earnings Ratio

Model II

In addition to defining the partial equilibrium conditions under which capital gains are likely to differ from retained earnings, it is also possible to specify the size of the difference as a function of the discount rate, the rate of return and the dividend payout ratio. Consider first what can be expected to happen if Model II is the model used by the market to evaluate the present value of an investment financed with retained earnings. In Model II the expected rate of return, the risk-adjusted discount rate, and the growth rate are all specific to the investment and a change
in capital value due to retained earnings is: ¹

\[ \Delta V = \frac{ar' \cdot R}{\beta'} - (1-a)r' \]

Suppose we let \( \Delta V = aR \) where \( a \) is the ratio by which the capital gain differs from retained earnings. ² If we substitute in (7) and solve for \( a \) we have:

\[ a = \frac{ar'}{\beta'} - (1-a)r' = \frac{a \cdot r'/\beta'}{1 - (1-a)r'/\beta'} \]

Unless \( a = 1 \) (i.e., all income paid as dividends), the ratio of gains to retained earnings, \( a \), will be different from the ratio of expected return to discount rate, \( r'/\beta' \). The relationship between the two ratios is shown in Figure IV for selected values of \( a \).

It is apparent that for low values of \( a \) (i.e., large retained earnings), a relatively small positive difference between the rate of return and the discount rate will produce large multiples of retained earnings in the form of capital gains. For example, if the dividend ratio is .25, a rate of return on investment that is 1/5 greater than the discount

¹ See Appendix A for a derivation of (7).

² \( a \) is also the coefficient on retained earnings in the regression model. The formula is valid only for \((1-a)r'/\beta' < 1\), i.e., the projected growth rate is less than the discount rate.
Figure IV: The gains / retained earnings ratio (a) as a function of return/discount rate ratio for selected values of a. (Model II)

\[ a = \frac{\alpha r'/\beta'}{1 - (1-\alpha)r'/\beta'} \]

(NOTE: Valid only for values such that \((1-\alpha)r'/\beta' < 1\) )
rate (i.e., \( r'/\beta' = 1.2 \)) will result in capital gains equal to three times the retained earnings invested. On the other hand, if the dividend payout ratio is .75, the rate of return would need to be twice the discount rate to create gains three times the value of retained earnings.

The graph indicates that the relationship between capital gains and retained earnings is extremely sensitive to the ratio of the expected return to the discount rate, especially for companies with large amounts of retained earnings. Thus, if a company consistently invests its retained earnings at an expected return greater than the cost of capital, investors can consistently expect that capital gains due to reinvestment will exceed the amount of earnings retained. In fact, values between 2 and 4 may well not be uncommon during periods when investors are optimistic and the discount rates they apply to earnings are low relative to the forecast rate of growth. On the other hand, the model implies that if discount rates rise, capital values will fall more than proportionately to the rise in the discount rate.

Model I

If the market uses Model I for evaluating the investment of retained earnings the results are similar to those just derived but sufficiently different to warrant some discussion.
Under Model I, the historical rate of return, rather than the expected rate of return on new investment, is used to estimate the rate of growth and the relationship between retained earnings and capital gains is given by equation (9):

\[
\Delta V_t = \frac{a r' R}{\beta' - (1-a) r_0}
\]

Once again, letting \( \Delta V_t = a R \) we have:

\[
a = \frac{a r'}{\beta' - (1-a) r_0}
\]

In order to evaluate the relationship between \( a \) and the ratio \( r'/\beta' \) we can set \( r_0 = b \beta' \) where \( b > 1 \). We then have:

\[
(a') = \frac{a r'/\beta'}{1 - (1-a)b}
\]

For given values for \( b \) and \( a \), \( a \) is a linear function of \( r'/\beta' \). In figure V are shown some of the possible relationships between \( a \) and \( r'/\beta' \) for specific values of \( b \) and \( a \). When one compares figure V to figure IV, it is quite evident the two models could yield much different values for the ratio \( a \) for a given value for \( r'/\beta' \).
Figure V: The Gains/Retained earnings ratio (a) as a function of the return/discount rate ratio \( r'/\beta' \) for selected values of \( b \) and \( a \). (Model II)

\[
a = \frac{ar'/\beta'}{(1 - (1-a)b)} \quad \quad b = \frac{r_0}{\beta'}
\]
C. The Regression Model and the Ratio $a$

An estimate of $a$, the ratio of capital gains to retained earnings, will be obtained directly from the regression equation as $a_1$, the coefficient on retained earnings. As we have indicated, the value of $a$ may vary considerably among firms, but it appears that most values should be positive and less than 4 or 5.\(^1\) Negative values for the coefficient on retained earnings are by no means ruled out and should be found for any firm whose total profits-- and stock prices-- consistently decline while new investment (positive retained earnings) is occurring, ceterus paribus. This would be interpreted by the model as a negative return to new investment. The numbers of such firms should be relatively small, however, since a firm with consistently negative returns on investment will eventually go out of business or be reorganized or taken over.

For an individual firm the value of the estimate of the ratio $a$ will be an average over the period of the observations. Since both the discount rate and the rate of return can vary independently one would expect that $a$ would vary as well over time. The degree of variation would depend upon the variability

\(^1\)This upper limit is purely judgmental and based on examination of Figure IV.
of the firm's profit stream, rate of return, and discount rate. This variation in \( a \) would affect the significance level of the estimate of \( a \). We would expect that the significance level for firms with highly fluctuating profits would be less than that for firms with stable profits.

D. The Ratio \( A \) and the Tobin \( q \)

In evaluating the behavior of the stock market and its effect on the desirability of investment, Tobin has developed a ratio he labels "\( q \)" which is the ratio of the market value of physical assets (based on their earnings and a discount rate) to their replacement costs.\(^1\) If the market value exceeds the replacement cost, it is presumably worthwhile to invest and vice versa. He has estimated the value of \( q \) for the total market for a number of years.

The value we are calling \( a \) and its estimate \( a_1 \) are similar in nature to \( q \) but on a firm level. To illustrate this, we can take the partial derivative of \( V \) with respect to retained earnings using Model II:

\[
\frac{\partial V}{\partial \beta} = \frac{ar'g'}{\beta' - g'}.
\]

If we multiply the numerator and denominator of the right-

hand side by the value of retained earnings, we have:

\[ \Delta V = \frac{\Delta V}{R} \Delta B \]

where \( \Delta V \) in this case is the market value of the investment and \( R \) is retained earnings. Thus, the regression coefficient on retained earnings is an estimate of the ratio of market value of investment to its cost and is the same kind of animal as Tobin's q.\(^1\)

It may be possible to obtain an estimate of \( a \) on an annual basis using cross section data for a given year for a number of firms. Such an estimate could be made for all or any subgroup of firms in the sample and the annual values for \( a \) compared to Tobin's q values for the same year. It is not likely the magnitudes would be the same but it is possible the directions of change would be.

2. Changes in the Discount Rate and the Rate of Growth

It is likely that the most influential variables among those creating capital gains or losses are the nominal discount rate and the nominal expected rate of growth of a company. We shall discuss these two items together because they are nearly impossible to isolate empirically and because their theoretical effects are very similar. As in the earlier section we shall discuss two different models whose distinguishing characteristics are the same as those spelled out

before. In Model I, the growth rate, \( g \), is presumed to be somewhat independent of the rate of return, \( r \). In Model II, the growth rate and the expected return are always linked by the equation \( g = (1-a)r \) so that the expected growth rate changes only if the rate of return does as well and

\[
\frac{\partial g}{\partial r} = (1-a).
\]

a. Model I

Suppose we begin with the basic equation:

\[
(2) \quad V = \frac{aE}{\beta - g} = \frac{arB}{\beta - g}.
\]

If we take the partials of \( V \) with respect to \( \beta \), \( g \), and their difference, \( \beta - g \), we have

\[
(11) \quad \frac{\partial V}{\partial \beta} = \frac{-aE}{(\beta - g)^2} = \frac{-V}{\beta - g},
\]

\[
(12) \quad \frac{\partial V}{\partial g} = \frac{aE}{(\beta - g)^2} = \frac{V}{\beta - g},
\]

\[
(13) \quad \frac{\partial V}{\partial (\beta - g)} = \frac{-aE}{(\beta - g)^2} = \frac{-V}{\beta - g}.
\]

Under the assumption that \( g \) and \( r \) can be independent, all three partials are identical except for sign. Thus, an observed change in the price of a share which is not accompanied by a change in current earnings, price level, or retained earnings, could be attributed to an identical change (except
for sign) in either the discount rate or expected growth, or to changes in both simultaneously. According to this model, if the expected rate of inflation were to affect both the growth rate and the discount rate equally, the inflation would cancel out and have no effect on the value of the stock. As we shall discuss later, this is a highly unlikely possibility.

b. Model II

If we drop the assumption that the expected rate of growth is independent of \( r \) and adopt the more reasonable hypothesis that \( g \) is a function of the rate of return on investment, we arrive at considerably different results from those obtained with the more simple model.\(^1\) In this case we have:

\[
\frac{\partial V}{\partial g} = \frac{\alpha B(\beta-g) \frac{\partial r}{\partial g} + (1-\alpha) \frac{\partial r}{\partial g} \alpha B}{(\beta-g)^2} = \frac{\alpha}{1-\alpha} \frac{B}{(\beta-g)} + \frac{V}{(\beta-g)}
\]

\[
\frac{\partial V}{\partial \beta} = -\frac{\alpha B}{(\beta-g)^2} = -\frac{V}{(\beta-g)}
\]

\[
\frac{\partial^2 V}{\partial (\beta-g)} = \frac{\alpha B}{(\beta-g)} \frac{\partial (\beta-g)}{\partial g} - \alpha B \frac{\partial}{\partial (\beta-g)} \frac{\partial}{\partial g} = \frac{\alpha B}{(\beta-g)} \frac{\partial r}{\partial g} = \frac{\partial r}{\partial g} \frac{B}{(\beta-g)} - \frac{V}{(\beta-g)}
\]

\[
\frac{1}{g} = (1-\alpha)r; \quad \frac{\partial r}{\partial g} = \frac{1}{1-\alpha}
\]
It is quite apparent that the symmetry found in Model I is no longer present. The partial with respect to $\beta$ is unchanged but the partial with respect to $g$ now consists of two terms, the second containing the effects of $g$ as it operates through the discount rate and the first containing the effects of a change in the rate of return on total profits. If the dividend payout ratio is low, the influence of the first term could exceed that of the second. In any case, a change in the rate of growth (reflecting a change in the rate of return) will nearly always have a larger effect than an equivalent change in the discount rate and could, in fact, be several times as large.

In looking at the third equation, the partial with respect to the difference between the discount rate and the growth rate, we see that the change in price will be a weighted average of the effects of the individual changes in $g$ and $\beta$ with the weights depending upon which changes most. Since we do not expect to be able to sort out the individual effects of growth and the discount rate, it is this equation that will concern us most.

The predicted effect of inflation in Model II is quite different from that in Model I. It is highly unlikely that the effects of inflation could simply cancel out, as they theoretically could in Model I. This is because a change in the growth rate depends upon a change in the rate of return
on investment. If a change in the expected rate of inflation does not affect the rate of return, \( r \), but does get incorporated into the discount rate, inflation will affect the value of a share only through the discount rate and the relationship between inflation and share price is almost certain to be negative.

If the rate of inflation is incorporated into both the discount rate and the rate of return, its effects on share price are still unlikely to cancel out. This can be easily seen if we compare the partial with respect to the rate of return to that for the discount rate:

\[
\frac{\partial V}{\partial r} = \frac{aB}{\beta - g} + \frac{(1-a)V}{\beta - g}
\]

(17)

\[
\frac{\partial V}{\partial \beta} = \frac{-V}{\beta - g}
\]

(15)

When \( r = \beta, \beta = V \) and the two equations are identical. In this case, if \( r \) and \( \beta \) were equally affected by inflation, a change in the rate of inflation would have no effect on share price. If \( r \) is not equal to \( \beta \), the share price might be positively or negatively affected by inflation depending upon whether \( r \) is less than or greater than \( \beta \). Of course, the most important question is whether \( r \) and \( \beta \) are likely to be equally affected by inflation but we shall defer discussion of this issue until later.
C. The Regression Model and the Discount Rate

Anyone who has worked with financial data for individual firms is aware of the difficulty of estimating either the market rate of discount for the firm's earnings or the expected rate of growth. Initial efforts will therefore be aimed at attempting to determine the role played in the creation of capital gains by the two together, i.e., changes in \((\beta-g)\). But direct estimates of \(\beta-g\) are just about as difficult to make as estimates of either of the two variables separately. It is possible, however, to calculate a value for the entire partial, defined either as equation (13) or (16).

The effect of a change in the difference between the discount rate and the growth rate \((\beta-g)\) on the price of a stock should be estimated holding everything else constant. Thus, if one could observe the price of a stock (in the same way one can observe the price of a bond) at the beginning and end of a period where absolutely nothing had changed about the firm other than the market discount rate and the expected growth rate, one could estimate the effect of such a change on price simply by subtracting one price from the other. Where such observations are not possible, however, it is necessary to adjust the second price observation for changes in earnings.

Thus, suppose we have beginning and end period values for a stock:
where $E_2$ might be affected by reinvested earnings, inflation or pure change. If we wish to eliminate the effects of earnings changes on $V_2$ we must essentially replace $E_2$ with $E_1$ and adjust the stock price to accommodate this replacement. The adjusted value for second period earnings is:

\begin{align*}
E_2^* &= E_2 \cdot \frac{E_1}{E_2} = E_1 \\
V_2^* &= \frac{aE_2^*}{(\beta-g)_2} = \frac{aE_1}{(\beta-g)_2}
\end{align*}

The estimated change in the value of the stock attributable to a change in discount rate/growth rate is:

\begin{align*}
\Delta V &= V_2^* - V_1 = \frac{E_1}{E_2} V_2 - V_1
\end{align*}

The value obtained in equation (19) is thus an estimate of the partial differential with respect to the difference between discount rate and growth.

This methodology implicitly takes into account the effects of inflation on the combination discount rate/growth rate, but it obviously does not distinguish between the effects inflation may have on each.

The variable defined in equation (19) is the variable to
be used in the regression equation (C) and is designated as \( X_2 \). Since the variable presumably approximates the partial derivative, the theoretical value of its coefficient is one.

3. **Expected Inflation and Price Level Changes**

Inflation is clearly one of the most important elements that must be dealt with in any model purporting to explain capital gains or relate them to other variables. It is essential in analyzing the effects of inflation, however, to distinguish between changes in the level of prices and changes in the expected rate of inflation because some variables are affected by one and some by the other. Nor are the two components of inflation linked in a predictable way; the level of prices may change continuously while the rate of inflation is constant.

In order to analyze the theoretical effects of inflation, we shall write our valuation equation in such a way that it includes both real terms and the price level. We shall also indicate which terms are functions of the rate of inflation. We thus have two inflation variables, the current price level, \( P \), and the rate of inflation, \( \dot{p} \). The price index is defined so that the current price level is always equal to 1.00. The real value of a firm's assets we shall define as \( B^* \). Equation (2) then becomes:
(2') \[ V = \frac{ar(p)B^*P}{\beta(p) - g(p)} \]

(a) Price Level Changes

If we take the derivative of (2') with respect to the price level we obtain:

\[ dV = \frac{ar(\dot{p})B^*dP}{\beta(\dot{p}) - g(\dot{p})} = VdP \]

We have not included any partials of variables that are functions of the inflation rate because (a) we wish to consider them separately and (b) there is no way to specify a priori how the expected rate of inflation will be affected by the current change in price level. The relationship may be positive, negative or zero depending upon the links between current price change and expectations. Thus, equation (20) does not comprehend the impact of expected inflation on the expected rate of return, the discount rate or the expected growth rate. These effects are quite separate from those of a simple one time change in prices.

The theoretical effects on share price depicted in (20) comes about because prices of both inputs and outputs rise, and as a result, the net profits rise as well. Alternatively stated, if the value of the firm's assets rises with inflation, while the rate of return and discount rate remain unchanged, the income stream and the firm's market value should go up.
as well.

There is no assurance that the firm's profits do respond to inflation in this manner, however. In fact, it is quite possible that inflation as it operates through the level of earnings (or equivalently, the value of assets) will not be fully reflected in stock price changes. All firms will not have the same ability to maintain the real value of their profit stream as the cost of inputs rise. How their profit stream changes as the average price level moves up will depend upon a number of things including the elasticity of demand for their products, the variety of goods produced, their debt/equity ratios and the maturities of their outstanding debt, and their capital/labor ratios. In addition to these operational characteristics, there are accounting regulations which affect both stated profits and cash flow. Depreciation, for example, must be figured on historical costs. During inflation, calculated depreciation understates true depreciation based on replacement costs, overstates profits, and results in excessive tax payments thus reducing cash flow. Undervaluation of inventory has the same effect and may be even more important than the understatement of depreciation.

It should be noted that the basic variable through which prices can be projected to operate is the market value of the stock itself. This makes sense, since in the absence of any changes, other than the effect of inflation on profits
or real asset value, the market value should rise by the degree of price increase. A valid alternative which will also be tried is earnings. Book value is not an adequate variable through which to measure the effects of price level changes.

(b) Changes in the inflation rate

We shall turn now to a consideration of the effects of changes in the rate of inflation on equation (2'). If we take the derivative with respect to the inflation rate (suppressing the price level which we said was 1.00 anyway) we have:

\[
dV = \frac{ar'B(\beta-g) - (\beta'-g')arB}{(\beta-g)^2} dp
\]

(21) \[dV = \frac{ar'Bdp}{\beta-g} - \frac{V(p'-g')dp}{\beta-g}\]

\[r' = \frac{dr}{dp} \quad , \quad g' = \frac{dg}{dp}\]

Utilizing our assumption that \( g = (1-\alpha)r \), (21) can be rewritten as:

(21') \[dV = \frac{ar'Bdp}{\beta - (1-\alpha)r} - \frac{V(\beta' - (1-\alpha)r')dp}{\beta - (1-\alpha)r}\]

If we accept the notion that inflation is incorporated in both the rate of return and the discount rate in the usual
Fisherian fashion and assume that \( \beta = r \) we have:

\[
\beta = \beta^* + \beta = r = r^* + \rho
\]

and

\[
\beta' = r' = 1
\]

(21") \( dV = \frac{aB\dot{p}}{a r} - \frac{Va\dot{p}}{a r} = 0 \) because \( B = V \) when \( \beta = r \).

The first term on the right-hand side of (21") is the change in value resulting from a change in expected rate of return due to inflation. The second term is the change in value due to a change in discount rate and growth rate. When the rate of return and the discount rate are equal, and the two respond identically to inflation, these two effects are precisely offsetting.

A more likely and therefore more interesting situation occurs when (a) the nominal rate of return and the discount rate are not equal and/or (b) when their responses to inflation are not identical. Let's look initially at the first of these two situations. Even though \( \beta \) and \( r \) are not equal, their response to inflation is assumed to be the same so that \( \beta' = r' = 1 \). In this case, (21) becomes:

(21") \( dV = \frac{aB\dot{p}}{\beta - (1-a)r} - \frac{aV\dot{p}}{\beta - (1-a)r} \).

The value of the equation hinges on the relationship
between $B$ and $V$ and this can be expressed as:

$$\frac{B}{V} = \frac{\beta - r}{ar} + 1 .$$

So that $B > V$ as $\beta > r$. If the discount rate exceeds the expected rate of return, an identical response on the part of both to inflation will lead to an increase in the price of the stock.

Suppose we now examine the second situation. What happens to price when the responses of the discount rate and the rate of return to a change in the rate of inflation are not identical, particularly when the rate of adjustment is different. Such a situation might well arise if the discount rate adjusts almost immediately to changed inflationary expectations but the expected rate of return does not. In this instance, $\beta' = 1$ and $r' = 0$ and (21) becomes:

$$dV = \frac{-Vdp}{\beta - (1-a)r} .$$

This equation is identical with (15), the partial with respect to $\beta$. It implies that an increase in the expected rate of inflation could have sizeable negative effects on the value of a stock if it were incorporated immediately into the discount rate while the expected rate of return were unchanged. Over time, if the expected rate of return gradually rose to accommodate the inflation rate, the loss in stock price
would be regained. If the inflation rate continued to increase, however, and there was always a lag in the rate of return, stock prices could remain depressed, or fall even more, until inflation leveled off or began to decline, at which time they should rise again. These effects would be independent of whether the discount rate and rate of return were equal initially.

It is quite possible that the phenomenon that lies behind equation (21) helps explain what has been happening to the stock market over the last few years. The discount rate has risen with inflation while the rate of return has lagged behind.

It should be emphasized that there is a major difference between the effects of inflation on capital gains as it operates through earnings and as it operates through the discount rate and projected growth rate. Its effects on earnings are real, observable and operational. They are ex post and it is realized inflation that is the important variable. Prices change and a firm's sales, costs and profits change with them. The firm may take certain actions in an attempt to anticipate the effects of inflation but the actual price changes are the events that affect earnings. The discount rate and the anticipated growth rate, on the other hand, are influenced primarily by expectations and are market phenomena. An expected increase in the rate of inflation should have an
immediate effect on the interest rate. How it will influence the projected growth rate will depend upon how future profits are expected to be changed by inflation but these expectations will also be incorporated immediately in valuing the income stream.

(c) The regression model and common stock as an inflation hedge

By now it should be evident that the impact of only one of the two inflation variables, changes in the price level, can be independently evaluated in our regression equation. This is because changes in the expected rate of inflation will affect the discount rate variable and be included in the effects measured there.

The variable that will be used in the regression equation is the partial defined in equation (20) which is the change in price level multiplied by the share price. It is designated as $X_3$ in equation (C). There is no particular value one should expect for the coefficient on this variable. If the firm has been able to hedge its profits against inflation should be positive. The coefficient should equal one if profits rise identically with inflation, be greater than one if the firm does better than inflation and less than one if it does worse. If the firm's stock prices have declined during inflationary periods, ceterus paribus, the coefficient should be negative.
We should emphasize that this definition of "inflation hedge" is somewhat different from that normally used by people looking at the ability of assets to act as hedges against inflation. But we think it is the appropriate definition for income earning assets. If the level of income rises so that the real level of income on an initial investment is unchanged for a stock holder (assuming he has been able to consume all previous real income) the asset has presumably been a good hedge against inflation. The capital value of the asset might well fall even though the real income level was unchanged, simply because of an increase in the discount rate due to inflation but this is a different aspect of the problem of hedging against inflation.¹

4. Unexpected Changes in Earnings

Two of the variables we have discussed thus far, retained

¹If it is assumed that inflation is nearly always unpredictable, the value of an asset as an inflation hedge should be measured by observing it not only during a period of inflation but before and after as well. In other words, how does it perform under a buy and hold strategy. Otherwise, its efficiency as an inflation hedge depends upon the astuteness of the individual making the investment decisions and his ability to (a) predict inflation and (b) buy those assets which perform best at any particular point in the inflationary cycle.
earnings and inflation, could be said to produce expected changes in earnings. Other expected changes will be reflected in the market's estimate of the growth rate. Nevertheless, there will always be unexpected events, political or economic in nature, which deflect the earnings stream from its expected path. They can never be predicted and their effects can only be observed ex post. Even ex post, however, it is extremely difficult to determine what portion of earnings can properly be defined as "expected" and what portion should be classified as "unexpected." In addition the influence of unexpected earnings changes will differ depending upon whether they are considered by the market to be permanent or temporary.

Defining a variable that one could call ex ante expected earnings is not something that can be done with great confidence. There are a number of simple methods that could be used such as projecting the historical growth rate or using projections made by Standard and Poors or some other investment advisory service. Alternatively one could use an econometric model that took into account the value of retained earnings, expected inflation, and any other variables that might influence earnings. A third method would be to use time series analysis in which a sequence of observations on a given variable is viewed as a realization of jointly distributed random variables. Finally, one could ignore as an explanatory phenomenon any variable called "unexpected earnings" and accept as
a measure of the effect of unexpected earnings changes the
residual variance of the regression equation; i.e., that part
of the capital gains which are not explained by the other
variables. This is, in fact, the method we shall use as
the basic case, to simply let the residual variance include
all effects not attributable either to changes in the discount
rate, retained earnings or inflation. Then the explanatory
quality of various definitions of unforeseen earnings changes
can be measured by evaluating how much of the residual variance
from the base case is eliminated by including a new variable.

A simple expression of the way in which an unexpected
change in earnings might be reflected in capital gains is
the following, where \( E(V_1) \) is the expected price one year
hence, taking into account the effects of other variables,
and \( V_1^* \) is the actual price adjusted for the effects of other
variables:

\[
E(V_1) = \frac{\alpha E(E_1)}{\beta - g}
\]

\( V_1^* = \frac{\alpha E}{\beta - g} \).

The effect on capital gains would be:

\[
(22) \quad V_1 - E(V_1) = \frac{\alpha [E_1 - E(E)]}{\beta - g}
\]

If the variable on the right-hand side can be appropriately
defined, its role in the formulation of capital gains could
be evaluated.

If the unexpected change in earnings were measured accurately and if it were considered permanent by the market, the change in stock price should reflect fully the future stream that the unexpected earnings change represents. If the change were to be interpreted as transitory, the change in value would be less, perhaps considerably less, than the full present value of the change. In the first case, the regression coefficient attached to the unexpected earnings variables (\(X_4\) in equation (1)) would be one and significant. If expectations are not well measured or if unexpected deviations from trend are considered transitory, the coefficient should be less than one and is much less likely to be significant.

**CONCLUSIONS**

The primary purpose of this paper has been to develop a simple model that lends itself to analysis of the relationship among capital gains on common stocks and the four factors that theoretically lie behind such gains: retained earnings, changes in the market discount rate, expected inflation and annual price changes, and the unexpected changes in earnings.

The model developed is composed of partial derivatives of the basic stock evaluation equation, \(V = \frac{\alpha E}{e^t - g}\).
\[ dV = \frac{\partial V}{\partial B} dB + \frac{\partial V}{\partial (\beta - p)} (\beta - g) + \frac{\partial V_t}{\partial P} dP + \frac{\partial V}{\partial E} dE \]

The right-hand side elements are, respectively, the effect of retained earnings, the effect of discount rate changes, the effect of price level changes, and the effect of changes in expected earnings.

A close examination of these derivations reveals a number of interesting results. First, the relationship between retained earnings and capital gains appears to hinge primarily on the ratio between the rate of return on investment and the discount rate applied by the market to the firm's revenue stream. For example, a ratio of 1.2 between rate of return and discount rate could lead to a ratio of capital gains to retained earnings of anything between 1.2 and 3.0, depending upon the firm's retained earnings ratio, with higher retained earnings leading to higher capital gains. Only if the discount rate and rate of return are equal would one expect gains to equal retained earnings. The ratio of capital gains to retained earnings has an interpretation that is similar to that for Tobin's q which is the ratio of market to replacement value for aggregate assets in the economy except that it is on a firm level.

Second, changes in the expected rate of growth and changes in the discount rate can be expected to have effects on stock prices that differ not only in sign but in magnitude.
Thus, if the expected growth rate and the discount rate were to increase identically, the price of a share should rise because the growth rate should increase only if the rate of return goes up.

In fact, it is really the rate of return rather than the growth rate that one should focus on because the rate of return determines the growth rate. An increase in rate of return matched by an increase in the discount rate could lead to a rise, a fall or no change in the price of a stock, depending upon whether the rate of return is less than, greater than or equal to the discount rate.

This has interesting implications for the effects of changes in the expected rate of inflation on stock price. Such expectations would presumably be incorporated almost instantaneously in the discount rate. Considerable uncertainty, however, might surround the effects of inflation on future profits and the expected growth rate might not reflect the changed inflationary expectations for some time. If there is such a lag in the rate of return and growth rate, an increase in the rate of inflation should lead to a fall in stock prices with the lower level persisting or perhaps falling even further, so long as the rate of inflation is increasing. If the expected rate of inflation declines, the movement of stock prices should be reversed.

Third, annual changes in the price level should, on
average, lead to equivalent percentage changes in the value of common stocks, if firms' adjustment processes permit their profits to rise with the inflation rate. This phenomenon should be independent of the effects of inflationary expectations on the discount rate and growth rate. It is this phenomenon alone that one should look at when evaluating whether common stocks are good hedges against inflation for by definition, a hedge is an asset whose earnings rise with inflation. The effects of inflation on the interest rate at which the earnings are evaluated and consequently upon the capital value of these earnings is another question altogether.
The model of stock valuation used in this study is the standard model often referred to as the Discounted Cash Flow (DCF) model and forms the basis of most analytical work on the valuation of uncertain streams of future income. It is analogous to the usual present value model of a certain stream of income such as that accruing to a bond. A number of assumptions are made which permit the present value of the uncertain stream to be treated as a function of current cash flow (typically dividends), a rate of growth for the cash flow and a discount rate which includes a risk premium specific to the firm. The basic relation is:

\[(1-A) \ V_t = V(D_t, \ b, \ g)\]

where
- \(V_t\) = current price of a stock
- \(D_t\) = \(\alpha E_t\) = current dividend level
- \(\alpha\) = dividend payout ratio
- \(E_t\) = current earnings
- \(b\) = rate at which the market discounts the firm's earnings stream
- \(g\) = projected rate of growth of income stream.
If $V_t$ is the present value of all future income, this relationship can be written as:

$$V_t = \int_0^\infty E_t e^{(\beta-g)q} dq$$

(2-A)

where $q$ is the time variable. Integrating, we get an equation completely analogous to that for a consol:

$$V_t = \frac{aE_t}{\beta-g}$$

(3-A)

Some of the elements in equation (3-A) can be broken down still further. Earnings, for example, can be expressed as the rate of return, $r$, times the value of the firm's assets, $B$. $E = rB$. The rate of growth can also be expressed in terms of the rate of return and the dividend payout ratio:

$$g = \frac{\Delta E}{E} = \frac{r\Delta B}{rB} = \frac{r(1-a)E}{rB} = r(1-a)$$

where $\Delta B$ is retained earnings. Thus, (3-A) can be written a number of equivalent ways, of which two are:

$$V_t = \frac{aE_t}{\beta-g} = \frac{arB_t}{\beta-(1-a)r}$$

(3-A)

Changes in the elements on the right-hand side of the equation result in changes in the market value of the stock

---

The value of a consol is $p_t = \frac{C}{r}$ where $C$ is the coupon rate, $r$ is the market discount rate, and $p_t$ the current price. The equation for a stock differs only in that the expected rate of growth, $g$, effects the value, increasing it if $g$ rises, decreasing it if $g$ falls.
and produce capital gains or losses. In order to formally evaluate how changes in each variable could be expected to influence stock prices, we can take the total derivative of equation (3-A) and see how the stock price can be expected to change as each of the right-hand variables fluctuates. Two variables that are implicit in equation (3-A) are the price level, \( P \), and the rate of inflation, \( p \).

\[
(4-A) \quad \Delta V_t = \frac{\partial V_t}{\partial \beta} \Delta \beta_t + \frac{\partial V_t}{\partial g} \Delta g + \frac{\partial V_t}{\partial p} \Delta p + \frac{\partial V_t}{\partial P} \Delta P.
\]

The terms on the right-hand side of equation (4-A) are the derivatives with respect to: (a) retained earnings, (b) the market discount rate, (c) the rate of growth, (d) the rate of inflation, and (e) the price level. Each of these terms will be analyzed to evaluate the potential influence of each of the variables.

**Discount Rate and Growth Rate**

We shall begin with the effects of changes in the discount rate and the growth rate. We treat these variables jointly because their effects are similar and it is difficult empirically to isolate the influence of one variable from that of the other. If \( \beta \) and \( g \) are independent of the other variables in the equation we have the following relationships. With respect to the discount rate:
With respect to growth:

\[(5-A) \quad \frac{dV}{\beta} = \frac{aE}{(\beta-g)^2} = \frac{-V}{\beta-g} \]

With respect to their difference:

\[(6-A) \quad \frac{dV}{g} = \frac{aE}{(\beta-g)^2} = \frac{V}{\beta-g} \]

Identical changes in \( \beta \) and \( g \), if neither is functionally related to the other variables, would cancel each other out and the net effect would be zero. This assumption of independence, however, is likely to be valid only for the discount rate. It should not hold for the rate of growth. We have in fact already postulated that growth is a function of the rate of return on investment which in turn determines earnings. If growth is expressed as \( g = (1-\alpha)r \) the differentials with respect to the discount rate, the growth rate, the difference between the two, and the rate of return can be written as:

**discount rate**

\[(8-A) \quad \frac{dV}{\beta} = \frac{-arB}{(\beta-g)^2} = \frac{-V}{(\beta-g)} \]

**growth rate**

\[(9-A) \quad \frac{dV}{g} = \frac{aB}{(\beta-g)(\beta-g)} + \frac{(1-\alpha)V}{(\beta-g)(\beta-g)} \]

if \( g = (1-\alpha)r \) and \( \frac{\partial r}{\partial g} = \frac{1}{1-\alpha} \)
\[
\frac{\mathrm{d}V_t}{\mathrm{d}g} = \frac{1}{(1-\alpha) (\beta-g)} + \frac{V_t}{\beta-g}
\]

difference between discount rate and growth rate

\[
\frac{\mathrm{d}V_t}{\mathrm{d}(\beta-g)} = \frac{\alpha B_t}{\beta-g} \frac{\partial (r)}{\partial (\beta-g)} - \frac{V_t}{\beta-g}
\]

if \(\frac{\partial r}{\partial (\beta-g)} = \frac{\partial r}{\partial g} \cdot \frac{\partial g}{\partial (\beta-g)}\) and \(\frac{\partial r}{\partial g} = \frac{1}{1-\alpha}\)

\[
\frac{\mathrm{d}V_t}{\mathrm{d}(\beta-g)} = \frac{\alpha B_t}{1-\alpha} \frac{\partial g}{\partial (\beta-g)} - \frac{V_t}{\beta-g}
\]

rate of return

\[
\frac{\mathrm{d}V_t}{\mathrm{d}r} = \frac{\alpha B_t}{\beta-g} + \frac{(1-\alpha)V_t}{\beta-g}
\]

When growth depends upon the rate of return, equal but opposite changes in the growth rate and the discount rate will not be offsetting. The growth rate has more impact than the discount rate. In any case, only very rare circumstances might produce an equivalent change in the growth rate and the discount rate. The two variables that are much more closely linked are the discount rate and the rate of return, \(r\). Although the links are not necessarily immediate and direct, economic theory would lead us to predict that if the cost of capital changes, the rate of return will eventually follow as the firm adjusts its investment behavior. At the firm level, the reverse in not likely to be true. A higher rate of return should not result
in a higher cost of capital. If the aggregate level of return for an economy rises, however, it may lead to a higher cost of capital for everyone if demand for investment funds expands.

If, however, in spite of the odds against it, one were to observe an equal rise in the rate of return and the discount rate, the change in share price might be zero, negative or positive, depending upon whether the book value were equal to, less than or greater than the market value. In general, one would expect that the discount rate would fluctuate much more often and more widely than expected rate of return (and with it the expected rate of growth) for the firm. Thus, most of the variation in share price that is due to the two of them together is probably due to changes in the discount rate.

One important factor that may or may not affect the discount rate but which will certainly affect the growth rate is the retained earnings ratio. Growth in earnings per share, particularly for an established corporation, is more likely to be linked to the reinvestment of earnings than to any other factor and it is the dividend payout ratio that determines the amount of retained earnings. Rewriting equation (3) with growth as an explicit function of the dividend payout ratio we have:

---

1An unresolved debate in the financial literature concerns whether dividend policy affects the cost of capital for a firm.
The differential with respect to \( a \) is:

\[
(3-A) \quad V_t = \frac{aE_t}{\beta - (1-a)r}
\]

(13-A) \[dV_t = \frac{E_t(\beta-g)da - raE_t da}{(\beta-g)^2} = \frac{E_t da}{\beta-g} - \frac{rV_t da}{\beta-g} = \frac{rB_t da}{\beta-g} - \frac{rV_t da}{\beta-g}.\]

We thus have the conditions:

\[
(14-A) \quad \frac{dV_t}{da} \geq 0 \quad \text{as} \quad B_t \leq V_t
\]

\[
\frac{dV_t}{da} < 0 \quad \text{as} \quad \beta \geq r.
\]

Although the influence on the growth rate of a change in the retained earnings ratio is predictable and positive, the instantaneous effect on the value of the firm is not. Economic theory implies that the profit maximizing firm will equate \( r \) to \( \beta \). If this is in fact the case, the value of the firm at a point in time will be unchanged by a change in the growth rate that results from a change in the dividend ratio. This is because the change in growth rate is exactly offset by the change in dividends. If, however, the rate
of return on retained earnings is below the cost of capital, an increase in retained earnings will cause the value of the firm to fall and vice versa, a not unsurprising result. In contrast to the instantaneous change, the time path of the firm's value over a period of time could be significantly affected by the retained earnings ratio even when the discount rate and rate of return on investment are equal. We shall discuss this point more fully when we look at the effects of retained earnings on capital gains.

The difficulty of separating the effects of changes in growth from those caused by movements in the discount rate leads us, at least initially, to discard consideration of the two separately and treat the difference between the two as a single variable. An important thing to note is that this variable, \( \beta - g \), is equal to the dividend yield for a stock, and that equation (11-A) is the differential with respect to the dividend yield. This is the nominal yield and therefore includes the expected rate of inflation whose effects we shall discuss more fully in the section on inflation.

Retained Earnings

If one wishes to examine the theoretical effects of retained earnings on stock price under the continuing assumption that the firm earns a rate \( r \) on its old and new investments, one can differentiate (3-A) with respect to
book value:

\[(15-A) \quad dV_t = \frac{\alpha r dB_t}{\beta - (1-\alpha)r} \quad R_t = dB_t = \text{retained earnings}.\]

We thus have the conditional relationship between capital gains and retained earnings of:

\[(16-A) \quad dV_t < R \quad \text{as} \quad \beta < r .\]

If, for a firm, the cost of capital is consistently greater than the return on investment, the capital gains attributable to retained earnings will be less than the earnings retained and stockholders would have been better off had they received dividends.\(^1\)

This derivation assumes that the dividend ratio remains constant but the same result is obtained if the dividend ratio is changing. This can be demonstrated by taking the differential with respect both to the change in book value and the retained earnings ratio:

\(^1\)In a multiple regression equation with capital gains as the dependent variable the coefficient on retained earnings should indicate whether or not the return on investment has been consistently below the cost of capital. If it has, the coefficient should be less than one. If return is consistently above the cost of capital, the coefficient should exceed one. Interpreting of the coefficient should shed some light on the efforts of Baumol....Whittington, Mueller, Grabowski, etc., to differentiate between return on retained earnings and return on outside capital.
(17-A) \[ \frac{dV_t}{dt} = \frac{\alpha dB_t}{\beta-(1-\alpha)\beta} + \frac{rB_t da}{\beta-(1-\alpha)\beta} - \frac{rB_t da}{\beta-(1-\alpha)\beta} \]

\[ = \frac{\alpha dB_t}{\beta-(1-\alpha)\beta} + \frac{Vda}{\alpha} - \frac{rVda}{\beta-(1-\alpha)\beta} \]

If \( \beta = r \), (17-A) becomes:

(17'-A) \[ dV_t = dB_t + \frac{Vda}{\alpha} - \frac{Vda}{\alpha} = dB_t \]

and capital gains are equal to retained earnings even though the retained earnings ratio may be changing.

The interaction between the dividend ratio, the discount rate and rate of return, and capital gains can be further evaluated by examining how a change in the rate of growth of retained earnings can be expected to influence share prices. Retained earnings can be expressed as \( R_t = (1-\alpha)B_t \). Thus \( dR_t = -rB_t da \) and \( da = -dR/rB \). If this expression is substituted into equation (17-A) we have:

\[ dV_t = \frac{\alpha dB_t}{\beta-(1-\alpha)\beta} - \frac{Vda}{\alpha} + \frac{rVda}{rB[\beta-(1-\alpha)\beta]} \]

(17''-A) \[ = \frac{\alpha dB_t}{\beta-(1-\alpha)\beta} + \frac{dR}{\beta-(1-\alpha)\beta} \left( \frac{V-B}{B} \right) \frac{V > B \text{ as } \beta > r}{V < B \text{ as } \beta < r} \]

If \( \beta = r \), \( V = B \) and we have the result obtained previously, i.e. \( dV_t = R \), and any increase or decrease in the rate of
accumulation will have no effect on this relationship. If, however, \( \beta < r \), capital gains will exceed retained earnings and an increase in the rate of accumulation will serve to accentuate this relationship. On the other hand, a decrease in the rate of accumulation of earnings will push the ratio of capital gains to reinvested earnings back toward one. Just the opposite effects should be observed if \( \beta > r \).

The above relationships were all derived under the assumption that the rate of return on existing capital and that on new capital was the same, that the market rate of discount applied to both earnings streams was the same, and that expected growth rates of both income streams were the same. None of these assumptions may hold, particularly for firms that diversify into new areas where expected growth rates and rates of return may be quite different from those attached to previous investments. Certain financial theories tell us that the cost of capital for a project should be a function of the risk and other characteristics of the earnings stream for that project rather than the historical cost of capital for the firm. If the rate of return, the discount rate, and the growth rate are specific to the investment project we have the following where \( \beta' \), \( r' \), and \( g' \) apply to the new project:

\[
V_t = \frac{\alpha r B_t}{\beta - (1-a) r} \quad \text{if } \beta = r.
\]
\[
V_0 = \frac{ar_0B}{\beta_0 - (1-a)r_0} \\
V_1 = \frac{ar_0B}{\beta_0 - (1-a)r_0} + \frac{ar'R}{\beta' - (1-a)r'}
\]

\[
(18-A) \Delta V_t = V_1 - V_2 = \frac{ar'R}{\beta' - (1-a)r'}
\]

This formula is in fact no different from (15-A) except that we have specifically allowed for the possibility that the rate of return, the market discount rate, and the rate of growth may all be different from those applied to the current capital and income streams of the firm. Let us assume that the market, on the basis of all currently available knowledge, assigns the appropriate discount rate to the new earnings stream, even though it may be different from that assigned the old one.¹ There are then two major possible alternative ways in which the market may evaluate the new earnings stream. We shall refer to these as Model I and Model II. Under Model I, the expected rate of growth of the earnings stream is assumed identical to that of the primary earnings stream, i.e., \( g' = g_0 = (1-a)r_0 \), so that:

\[
(18'-A) \Delta V_t = \frac{ar'R}{\beta' - (1-a)r_0}
\]

The condition for equality of retained earnings and capital

¹The efficient market hypothesis and capital market theory imply this should be the case.
The gains under Model I is:

\[(19\text{-A}) \quad \beta' = r_0 + a(r' - r_0)\]

or

\[(19'\text{-A}) \quad r' = r_0 + \frac{\beta' - r_0}{a} = \frac{\beta' - g_0}{a}\]

Thus, if expected growth is determined by the rate of return on existing capital, equality between the discount rate and the rate of return on new investment will no longer produce gains equal to retained earnings. If \(\beta' = r_0\) or if \(r_0 = r'\), the usual condition is reestablished which calls for \(\beta' = r'\). The inequality conditions are as follows:

\[\Delta V_t > R \quad \text{as} \quad \beta' > r_0 + a(r' - r_0)\]

and

\[\Delta V_t < R \quad \text{as} \quad r' < r_0 + \frac{\beta' - r_0}{a}\]

Model II is one in which investors apply a rate of growth to the income stream from new investment that may be different from the growth rate applied to the old stream. The growth rate is a function of the new expected rate of return, i.e., \(g' = (1-a)r'\), and the relationship between gains and retained earnings is given in equation (18-A):

\[(18\text{-A}) \quad \Delta V_t = \frac{ar'R}{\beta' - (1-a)r'}\]

Equality of retained earnings and capital gains is attained
when $\beta' = r'$.

This is the usual condition except that the discount rate and the return are both specific to the marginal investment undertaken with the retained earnings. The inequality condition is $^1$:

$$\Delta V_t \geq R \text{ as } \beta' \geq r' .$$

In summary, it is evident that the necessary condition for retained earnings to result in capital gains equal to the amount of retention is that the capital market discount rate for the stock be equal to the rate of return on investment. It is unlikely that this condition will be fulfilled at a given moment in time but, since it is presumably the principle that guides the investment decisions of firms, it may well be that over time and on average it is achieved. One of the main objectives of this study is to determine whether this occurs.

Inflation

It has often been assumed that common stocks are a good hedge against loss of purchasing power. If, on average, prices of inputs and consumption goods rise at the same rate,

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$^1$The implications of both models are discussed more fully in the main body of the paper, pages 19-37.
and aggregate real output and sales are unchanged, total profits should rise at the same speed as the price level. Inflation affects more than just the profit level, however. It also influences the discount rate and the expected rate of growth. Thus, it is essential to distinguish between realized changes and changes in the expected rate of inflation.

To analyze these effects, we need to write equation (3-A) so that it includes the real value of assets, $B^*$, as well as the price level. We shall select our price index in such a way that the current price level, $P_t$, is always equal to 100. $\hat{p}$ is expected rate of inflation.

$$V_t = \frac{\alpha_r(\hat{p}) B^* P_t}{\beta(\hat{p}) - g(\hat{p})}$$

We shall first take the derivative with respect to the price level:

$$dV_t = \frac{\alpha_r(\hat{p}) B^* dP_t}{\beta(\hat{p}) - g(\hat{p})} = V_t dP_t$$

A realized change in the price level, to the extent it affects profits or the replacement value of capital (with the rate of return unchanged), should increase the value of a

---

1By real we simply mean dollar values expressed in terms of prices obtaining at the point in time when the price index is equal to 100.
share by the price increase times the share value. This variable is the measure of whether stocks are a good hedge against inflation and is independent of the effects of the expected rate of inflation on either the discount rate or the rate of growth.

The effects of changes in the expected rate of inflation on share value are quite different from those that might result from a direct change in the price level. Although expectations will be influenced by a price level change, there is no predictable relationship between the two. A given increase in price might lead to a higher, lower or unchanged expected rate of inflation, depending upon what had gone before. For this reason, the expected rate of inflation must be analyzed separately from the price level. If we take the derivative of (3'-A) with respect to the inflation rate we have:

$$\frac{dV_t}{dp} = \left[ aB_t(\beta-g)\frac{dr}{dp} - \frac{dg}{dp} - a\frac{dr}{dp} \right] \frac{dB_t}{dp} \frac{dp}{dp}$$

Utilizing the assumption that $g = (1-a)\beta$ this can be re-written as:

$$(21-A) \quad \frac{dV_t}{dp} = \frac{a\frac{dr}{dp}}{\beta-g} - \frac{V_t(\frac{dg}{dp} - (1-a)\frac{dr}{dp})}{\beta-g}$$

If the rate of return and the discount rate are equal and if the derivatives of each with respect to the rate of inflation are equal, the value of (21-A) is zero. If the
derivatives are equal but the discount rate and rate of return are not, (21-A) becomes:

\[ (21'-A) \quad dV_t = \frac{\alpha B \frac{dp}{dt}}{B-g} - \frac{\alpha V \frac{dp}{dt}}{B-g}. \]

If \( r > \beta \), \( \frac{dV_t}{dp} > 0 \). A rate of return greater than the discount rate implies \( B \) is larger than \( V \) and the price will increase with an increase in the expected inflation rate.

The result in (21'-A) assumes, however, that the discount rate and the rate of return both respond equally to a change in expectations about the rate of inflation. A much more likely event is that the discount rate responds but the rate of return does not, or if it does respond there is a significant lag. If we take the extreme case where \( \frac{d\beta}{dp} = 1 \) and \( \frac{dr}{dp} = 0 \), equation (21-A) becomes

\[ (21''-A) \quad dV_t = \frac{-V \frac{dp}{dt}}{B-g}. \]

The implication of (21''-A) (which is identical to equation (8-A), the differential with respect to the discount rate) is that an increase in the expected rate of inflation could have large negative effects on the price of a stock if expectations are immediately incorporated into the discount rate but do not affect the rate of return immediately. A continual increase in the rate of inflation could lead to
a continuous decline in the level of stock prices. It is quite possible that this effect explains a large part of the fall in stock prices that has occurred in the years since inflation became an important aspect of the economy.

Since the inflationary expectations operate through the nominal growth rate and the nominal discount rate, they are nearly impossible to isolate for observation. Thus, empirically, the effects of changes in expected inflation will be included in the effects on stock prices that are attributed to the discount rate and growth rate.

Unexpected Changes in Earnings

There is one final set of relations that closes the system and links the retained earnings to the future earnings stream and the unexpected changes in earnings. It is not possible to derive this function by taking a partial derivative of equation (3-A), with respect to earnings, however. This is because we have already taken the partial with respect to the two components of earnings, r and B, and thus exhausted the possible linkages between predicted earnings and the prices of shares. If we redefine equation (3-A) as a stochastic rather than deterministic model, however, we introduce a random element that comprehends the effects of all the variables not specifically included in (3-A). Among these variables are unexpected changes in earnings. For the sake
of comparability we can define this unexpected change in earnings as $dE_t$ and write:

$$dV_t = \frac{\alpha dE_t}{(\beta-g)}$$-(21-A)-

If the term $dE_t$ were to include all changes in earnings, it would also comprehend the effects of retained earnings. This is why it is defined as unexpected changes in earnings, i.e., those not attributable to the new investment. There are a number of ways it could be defined, all of which involve calculating an expected earnings variable and then subtracting it from actual earnings. Producing a variable that accurately produces expectations is most difficult, however, and it could equally well be argued that unexpected earnings cannot be properly defined and should properly be left as part of the residual variance.

A Complete Model

In the previous sections we have examined the effects of changing individually each of the independent variables on the right-hand side of the equation:

$$V_t = \frac{\alpha \beta^{t-p}}{t-t}$$-(3-A)-

In order to develop a model that lends itself to empirical evaluation it is only necessary to take the total differential
of equation (3-A) (remembering only that $dE_t$ is something of an adjunct and is the unexpected change in earnings rather than total change):

$$dV_t = \frac{\partial V_t}{\partial (\beta-g)} d(\beta-g) + \frac{\partial V_t}{\partial B_t} dB_t + \frac{\partial V_t}{\partial P_t} dP_t + \frac{\partial V_t}{\partial E_t} dE_t$$

This formulation converts quite naturally into a regression model once the variables are defined. There are two possible forms the definitions could take.

**Regression Equation: Form I**

First, if we were interested only in determining whether there was a significant relationship between capital gains and each of the elements on the right-hand side of equation (3-A) we could simply define the variables as:

$$x_1 = d(\beta-g) = \text{a change in the discount rate (or dividend yield)}$$

$$x_2 = dB_t = R_{t-1} = \text{retained earnings}$$

$$x_3 = dP_t = \text{a change in prices}$$

$$x_4 = dE_t = \text{unexpected change in earnings}$$

$$y_1 = dV_t = \text{capital gains}$$

and set up the equation:

$$y_1 = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \epsilon_1$$
The equation could be evaluated for its explanatory power and the coefficients tested for sign and significance. The absolute values of the coefficients would also give some indication of the values of the partial derivatives attached to each variable. These values may turn out to be of interest, but a second formulation of the regression equation is likely to be of more value.

**Regression Equation: Form II**

In the second formulation we define as the independent variable the combination of the partial and the change in the simple variable. Thus we have:

\[ x_1' = \frac{\partial V_t}{\partial (\beta-g)} d(\beta-g) = \frac{AE_t d(\beta-g)}{(\beta-g)^2} = \text{discount rate} \]

\[ x_2' = \frac{\partial V_t}{\partial \beta_t} dB_t = \frac{ar}{\beta-(1-a)r} dB_t = \text{retained earnings} \]

\[ x_3' = \frac{\partial V_t}{\partial P_t} dP_t = \frac{AE_t dP_t}{\beta-g} = \text{price level} \]

\[ x_4' = \frac{\partial V_t}{\partial E_t} dE_t = \frac{AE_t dE_t}{\beta-g} = \text{unexpected earnings} \]

\[ y_1 = dV_t \quad = \text{capital gains} \]

The regression equation is then:
Using these definitions, all the $a$'s fall within predictable ranges that are different for each. Certain conclusions can be drawn based upon the coefficient values. For example, if $a_1$ is less than one, it would indicate that retained earnings do not produce an equivalent amount of capital gains. In most cases this is likely to mean that $r < \beta$. The opposite would be true if $a_1$ is greater than one.

We should emphasize that the unexpected earnings variable is not essential to the equation and will only be retained if it can be satisfactorily specified.

The regression results can also be used to estimate the relative importance of the various sources of capital gains. This can be done in two ways. First, if the $R^2$ is high, the sum of terms on the right-hand side of equation (16-A) should be approximately equal to the total of capital gains, making allowances for differences in signs. Thus, one could measure the relative importance of each of the variables by first multiplying the mean of the absolute values of each by its coefficient to obtain an estimate of the total contribution of each variable. This figure, when divided by the sum of the absolute values of all contributions (including that of the random term) would provide an estimate of the proportion of the total gain attributable to that variable.
The second way of estimating the importance of each variable is to use a step-wise regression technique and then calculate the marginal amount of variance explained by each variable as it enters. This latter method has some merit as it takes into account the significance of coefficient estimates.

**Specification of Variables: Form I**

Specification of the variables for Form I of the regression equation is relatively straightforward. In tabular form they would be as follows:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Form</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>X₁'</td>
<td>d((g-g))ₜ</td>
<td>The change in the dividend rate between the beginning of period t and the end of period t.</td>
</tr>
<tr>
<td>X₂'</td>
<td>dBₜ = Rₜ₋₁</td>
<td>Retained earnings during period t-1.</td>
</tr>
<tr>
<td>X₃'</td>
<td>dPₜ</td>
<td>Change in price level in percentage terms between the beginning of period t and the end of t.</td>
</tr>
<tr>
<td>X₄'</td>
<td>dEₜ</td>
<td>Unexpected change in earnings. The difference between expected earnings and actual earnings during period t. Various definitions of unexpected earnings will be tested.</td>
</tr>
<tr>
<td>y</td>
<td>dVₜ</td>
<td>Capital gains during period t.</td>
</tr>
</tbody>
</table>
Specification of Variables: Form II

Specification of variables for Form II is a bit more complicated than for Form I since it involves not only the variables, but an estimate of the partial derivative. The estimate assumes the market discount rate, \( \beta \), equals the return on investment, \( r \).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Form</th>
<th>Accounting Specification</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>( \frac{\partial V_t}{\partial (\beta-g)} )</td>
<td>( \frac{E_t}{E_{t+1}} )</td>
<td>Change in capital value due solely to change in discount rate.</td>
</tr>
<tr>
<td>( X_2 )</td>
<td>( \frac{\partial V_t}{\partial B_t} )</td>
<td>( R_{t-1} )</td>
<td>Retained earnings during period ( t-1 ).</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>( \frac{\partial V_t}{\partial P_t} )</td>
<td>( V_t dP_t )</td>
<td>Stock price at the beginning of period ( t ) times the price change over period ( t ). (Stock price here can be replaced by earnings in alternative formulations.)</td>
</tr>
<tr>
<td>( X_4 )</td>
<td>( \frac{\partial V_t}{\partial E_t} )</td>
<td>( \frac{V_t}{E_t} dE_t )</td>
<td>The price earning ratio times the unexpected earnings change. (A number of alternative formulations will be used to estimate expected earnings.)</td>
</tr>
</tbody>
</table>

The account specifications for each of these variables is
derived as follows:

\[ X'_1 = \frac{\partial V_t}{\partial (\beta-g)} d(\beta-g) = \frac{\partial E_t d(\beta-g)}{(\beta-g)^2} = \frac{\partial E_t [(\beta-g)_{t+1} - (\beta-g)_t]}{(\beta-g)^2} \]

\[ = V_t \left[ \frac{(\beta-g)_{t+1}}{\beta-g_t} - 1 \right] \]

\[ = V_{t+1} E_{t+1} - V_t \]

\[ X'_2 = \frac{\partial V_t}{\partial B_t} dB_t = \frac{\alpha r}{\beta - (1-a)r} dB_t = R_{t-1} \text{ when } \beta = r \]

\[ X'_3 = \frac{\partial V_t}{\partial P_t} dP_t = \frac{\alpha E_t}{\beta-g} dP_t = V_t dP_t = B_t dP_t \text{ if } \beta = r \]

\[ X'_4 = \frac{\partial V_t}{\partial E_t} dE_t = \frac{\alpha dE_t}{\beta-g} = \frac{V_t}{E_t} dE_t \]

---

\[ X'_1 \text{ can also be derived as follows:} \]

\[ V_t = \frac{\alpha E_t}{(\beta-g)_t} \quad V_{t+1} = \frac{\alpha E_{t+1}}{(\beta-g)_{t+1}} \quad V^*_{t+1} = \frac{\alpha E_t}{(\beta-g)_{t+1}} \]

\[ X'_1 = V_{t+1}^* - V_t = V_{t+1} \frac{E_t}{E_{t+1}} - V_t \]
The Measurement Period

The period over which capital gains and the other variables are to be measured is arbitrary. One could measure the gains for one year, three years, five years or twenty, and attempt to determine how the gains relate to the other variables when measured over the same period. If one is doing a time series analysis of a single company the upper limit to the time span is determined by the number of observations needed to do the regression. It is not evident what will happen to coefficient estimates when different time spans are used. One might predict that retained earnings would become a more significant determinant of capital gains as the time span lengthened and the effects of other, more transitory variables weakened. But the time series estimates will be estimates of the averages over the entire observation period and they may be relatively little affected by the choice of time span.

Although much of the analysis will be time series on individual companies, we shall also do some cross-section regressions for given years among different companies grouped by industry, size, and other characteristics. In the cross-sections we shall be able to use any time space we wish that is less than 20 years. We shall also be able to evaluate the stability of the time series estimates of various coefficients.